Realisation of Lie algebras and of representations of Lie groups in terms of harmonic oscillators

1985 J. Phys. A: Math. Gen. 18 2863

(http://iopscience.iop.org/0305-4470/18/15/016)

J. Phys. A: Math. Gen. 18 (1985) 2863-2891. Printed in Great Britain

Realisation of Lie algebras and of representations of Lie groups in terms of harmonic oscillators†

T Fulton

Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218, USA

Received 11 December 1984, in final form 24 April 1985

Abstract. The algebras and irreducible representations (irreps) of compact Lie groups, including the exceptional groups, are realised in terms of sets of Bose oscillator (SHO) creation and annihilation operators. In particular, not only the tensor irreps, but also the spinor irreps of orthogonal groups of all ranks can be constructed using Bose (rather than the more customary Fermi) oscillators.

1. Introduction

Realisations of specific simple compact Lie algebras and at least some of the irreducible representations (irreps) of the corresponding groups in terms of Bose and/or Fermi oscillators have been extensively considered in the past, and have been found to be a useful tool in the analysis of a large class of physical problems. In particular, Schwinger (1965) has studied the generators and irreps of SU(2) in terms of two Bose oscillators (hereafter abbreviated SHO) to deal with angular momentum. These considerations have been extended (see Kramer and Moshinsky 1968) to U(n), and have found extensive applications in nuclear physics. Alternatively, the algebras of the classical groups (the unitary, orthogonal and symplectic groups A(n); B(n), D(n); C(n)) have been realised in terms of Fermi oscillators. The spinor irreps of the orthogonal groups have been constructed in this scheme and have had extensive and successful applications in the study of many electron atoms (Judd 1967, 1968, Wybourne 1974) and in particle physics (Casalbuoni 1980, Casalbuoni and Gatto 1980).

The oscillator technique has also been fruitfully used in the case of specific non-compact groups, U(1, 1) in particular, most recently by Alhassid *et al* (1983). They study the relationship of bound and continuum states and energy band structures to each other for various one-dimensional problems, and to related non-periodic and periodic potentials.

A uniform realisation of the algebras of all classical Lie groups and of their tensor irreps is considered by a number of authors (Gilmore 1974, Quesne 1973, Lohe and Hurst 1971, 1974, Exner *et al* 1976). All the references mentioned so far possess the common characteristic of having bilinear forms for the group generators. They are, in some papers, structures $a_i^\dagger a_j$ where $a_i^\dagger$ and a_j are SHO (or Fermi oscillator) creation and annihilation operators with appropriate hermiticity relations—$(a_i)^\dagger = a_i^\dagger$—and

† Supported in part by the US National Science Foundation.

commutation (anticommutation) relations. Alternatively (and preferentially) in the papers dealing systematically with the classical algebras and their irreps, the bilinear forms are constructed from p_i and q_j.

The spinor irreps of the orthogonal groups are the only ones which, to our knowledge, have been constructed for these groups using Fermi oscillators. These irreps involve various numbers of Fermi oscillator creation operators $a_i^\dagger$, one $a_i^\dagger$ for each index i, for different elements of the same spinor irrep. Spinor irreps have also been constructed by Lohe and Hurst (1973) and by Lohe (1973), using the formalism of the boson calculus. For groups of low rank, the representation space of the covering group is used. More generally, these representations are constructed in spaces of harmonic homogeneous polynomials, by finding new realisations of the Lie algebras of the orthogonal groups. The method appears to be somewhat cumbersome in applications.

There are other, nonlinear, realisations of Lie groups in terms of SHOs, specifically the SU(2) generators (Dyson 1956). This realisation is inconvenient for constructing irreps.

The list of work referred to above is not even an attempt at an exhaustive survey of a very extensive literature. Further references to other work can be found in the papers already quoted.

The points of departure of the present paper are the previous papers using SHO creation and annihilation operators. The work is entirely restricted to simple compact Lie groups. A unified approach is taken with respect to all groups. Generators and irreps are realised entirely in terms of Bose oscillators, even for spinor irreps. Our point of view grows from that of Dynkin (1950, 1957), of McKay and Patera (1981) and McKay *et al* (1977), as further developed in a collaborative effort by the author and two colleagues (Feldman *et al* (1981, 1984a, b), the last two hereafter called FFMa and FFMb, respectively). The emphasis of this latter work is to carry out the Dynkin–Patera approach in an orthogonal root and weight space (with small and non-essential modifications required in the case of A(n) and G(2)). Such an approach leads to a concise and simplified treatment, with expressions which are more general and algebraically explicit (e.g. for the phases of structure constants (FFMa)) than those arising from weight and root spaces expressed in terms of a non-orthogonal basis. The practical usefulness of this approach is demonstrated in FFMa and FFMb and will be further justified in what follows. The notation of FFMa and FFMb will be used in general in the present paper. Extensive use will also be made of the results of these references, but, in the interests of brevity, they will often only be referred to, rather than repeated here.

The stage for the detailed discussion is set below in § 2, in which much of the notation used later is defined and the group algebras and irreps are written down in terms of it.

Section 3 is devoted to the realisation of the generators and of all the irreps of each classical group in terms of SHO operators. For purposes of illustrating the approach, particular emphasis is placed on the derivation of the results for the groups $D(n) \equiv SO(2n)$; a briefer treatment, emphasising only the results, is given for the other classical groups: B(n), C(n) and A(n).

We must emphasise that we have nothing new to offer in the SHO realisations of the classical Lie algebras and nothing essentially new in the SHO realisations of their tensor irreps. They are treated briefly in order to illustrate further our notation and to motivate the changes in the formalism required to accommodate spinor irreps of the orthogonal groups in our version of SHOs. This development is also the point of departure for the analysis of the exceptional groups in the next section.

In § 4, the SHO approach is extended to the realisation of the generators and irreps of the exceptional groups.

A brief summary and discussion of the salient aspects of the results is presented in § 5.

2. Definitions and general considerations

We shall begin by expressing all simple Lie algebras in the Dynkin basis (Dynkin 1957, FFMa, FFMb) in the form

$$[\boldsymbol{H}, \boldsymbol{H}]=0, \tag{2.1}$$

$$[\boldsymbol{\alpha}(p)\cdot\boldsymbol{H}, E_{\boldsymbol{\alpha}(q)}]=\boldsymbol{\alpha}(p)\cdot\boldsymbol{\alpha}(q)E_{\boldsymbol{\alpha}(q)}, \tag{2.2}$$

$$[E_{\boldsymbol{\alpha}(p)}, E_{-\boldsymbol{\alpha}(p)}]=\boldsymbol{\alpha}(p)\cdot\boldsymbol{H}, \tag{2.3}$$

$$[E_{\boldsymbol{\alpha}(p)}, E_{\boldsymbol{\alpha}(q)}]=N_{\boldsymbol{\alpha}(p),\boldsymbol{\alpha}(q)}E_{\boldsymbol{\alpha}(m)}, \tag{2.4}$$

where the structure constant $N_{\boldsymbol{\alpha}(p),\boldsymbol{\alpha}(q)}$ is non-vanishing for

$$\boldsymbol{\alpha}(m)=\boldsymbol{\alpha}(p)+\boldsymbol{\alpha}(q) \tag{2.5}$$

a root.

The vector $\boldsymbol{H}$ has n components, where n is the rank of the algebra; the operators $E_{(\pm)\boldsymbol{\alpha}(p)}$ are step up (step down) operators in weight space.

The magnitude of the structure constants (Wybourne 1974, Carter 1972) is the same for all groups with roots of the same length (A(n), D(n), E(n)), and also for B(n). There are two magnitudes for the structure constants of C(n), F(4) and G(2). The phase factor of $N_{\boldsymbol{\alpha}(p),\boldsymbol{\alpha}(q)}$ is defined as

$$\exp(\mathrm{i}\Phi(p, q))\equiv N_{\boldsymbol{\alpha}(p),\boldsymbol{\alpha}(q)}/|N_{\boldsymbol{\alpha}(p),\boldsymbol{\alpha}(q)}|. \tag{2.6}$$

Detailed commutation relations, taking into account the magnitude of $N_{\boldsymbol{\alpha}(p),\boldsymbol{\alpha}(q)}$, have been written down for each specific type of simple compact Lie group in FFMa. These commutation relations also serve to define the various phase factors. They lead to Jacobi identities (Carter 1972) which, in turn, can be expressed as algebraic conditions on the phases. These conditions are displayed in FFMa. They do not determine unique phase choices. A particular solution for each phase is obtained in FFMa. All phases in these solutions are 0 or π:

$$\exp(\mathrm{i}\Phi)=\pm 1. \tag{2.7}$$

We shall adopt these phase choices in what follows and express our results in terms of them.

The generators $\boldsymbol{H}$ are Hermitian. In addition, we impose the physically convenient restriction

$$E_{-\boldsymbol{\alpha}(p)}=E^{\dagger}_{\boldsymbol{\alpha}(p)}. \tag{2.8}$$

We have a set of basis vectors, $\boldsymbol{\mu}_p$, in the orthonormal root and weight space such that

$$\boldsymbol{\mu}_p\cdot\boldsymbol{\mu}_q=\delta_{pq}. \tag{2.9}$$

We define the notation

$$\boldsymbol{\mu}_{|p|} \equiv (|p|), \tag{2.10}$$

$$-\boldsymbol{\mu}_{|p|} \equiv \boldsymbol{\mu}_{-|p|} \equiv (-|p|), \tag{2.11}$$

$$\tfrac{1}{2}\boldsymbol{\mu}_{|p|} \equiv (\tfrac{1}{2}|p|). \tag{2.12}$$

The normalisation condition for $\boldsymbol{\mu}_p$ in (2.9) differs from the treatment in FFMa, FFMb, so that we may avoid the later explicit appearance of the scale factor, s (defined in equations (4.13) and (4.17) of FFMb), in expressions for the generators. For A(n), we will need the set of non-orthogonal vectors $\boldsymbol{\chi}_p$:

$$\boldsymbol{\chi}_p \equiv \boldsymbol{\mu}_p - \frac{1}{n+1}\sum_{q=1}^{n+1} \boldsymbol{\mu}_q, \qquad p = 1, \ldots, n+1, \tag{2.13}$$

with

$$\sum_{p=1}^{n+1} \boldsymbol{\chi}_p = 0. \tag{2.14}$$

For G(2) we will need $\boldsymbol{\chi}_p$, with $p = 1, 2, 3$ as defined in (2.13). The roots for the classical groups are given in table 1, and those for the exceptional groups in table 2 of FFMa. (For $\boldsymbol{\lambda}_p$, appearing in these tables, substitute $\boldsymbol{\mu}_p$, defined in (2.9) above.) Using this form of the root, we can simplify our notation for $E_{\boldsymbol{\alpha}}$:

$$E_{\boldsymbol{\mu}_p - \boldsymbol{\mu}_q} \equiv E_{(p-q)} \tag{2.15}$$

for a typical 'vector' root and

$$E_{\frac{1}{2}(-\boldsymbol{\mu}_p + \boldsymbol{\mu}_q + \boldsymbol{\mu}_r + \boldsymbol{\mu}_s)} \equiv E_{\frac{1}{2}(-p+q+r+s)} \tag{2.16}$$

for a typical 'spinor' root. (The example given in (2.16) is for a generator of F(4).) Such roots with 'spinor' weights (see FFMa) appear in the exceptional groups E(8), E(7), E(6) and F(4). In terms of the definition, (2.15), we have the identity

$$E_{(p-q)} = E_{(-q+p)}, \tag{2.17}$$

since

$$\boldsymbol{\mu}_p - \boldsymbol{\mu}_q = -\boldsymbol{\mu}_q + \boldsymbol{\mu}_p, \tag{2.18}$$

and similar identities for spinor roots.

The notation for irreps is in terms of the Dynkin–Patera indices and the weight vector, $\boldsymbol{\Lambda}$, of the individual irrep element. For a given irrep element we have

$$\boldsymbol{H}|\boldsymbol{\Lambda}, [\mathcal{T}]\rangle = \boldsymbol{\Lambda}|\boldsymbol{\Lambda}, [\mathcal{T}]\rangle, \tag{2.19}$$

$$E_{\pm\boldsymbol{\alpha}(p)}|\boldsymbol{\Lambda}, [\mathcal{T}]\rangle = C(\boldsymbol{\Lambda}, \pm\boldsymbol{\alpha}(p), [\mathcal{T}])|\boldsymbol{\Lambda} \pm \boldsymbol{\alpha}(p), [\mathcal{T}]\rangle, \tag{2.20}$$

and C is a numerical function of its variables. The scalar irrep will be the 'vacuum state' and will be denoted by

$$|\mathbf{0}, [0]\rangle \equiv |0\rangle, \tag{2.21}$$

where [0] is the Dynkin–Patera symbol

$$[0] \equiv (0, 0, \ldots, 0).$$

Since most of the Dynkin–Patera symbols we will use will have a large number of zero entries, we will shorten the notation by denoting only the non-zero entries (except for the case of [0]) and indicating their position by a subscript. Thus, we shall take, for example

$$(1,0,0,0)\equiv[1_1], \qquad (0,1,0,0)\equiv[1_2], \qquad (2,0,0,0)\equiv[2_1], \qquad (1,0,0,1)\equiv[1_1, 1_4]. \tag{2.22}$$

The SHO creation and annihilation operators will be denoted by

$$a^{(\kappa)\dagger}(\mathscr{L}) \qquad \text{and} \qquad a^{(\kappa)}(\mathscr{L}), \tag{2.23}$$

where (κ) and $(\mathscr{L})$ are a single index and a set of labelling indices respectively. The operators have the usual commutation relations

$$\begin{aligned}
&[a^{(\kappa)}(\mathscr{L}), a^{(\kappa')\dagger}(\mathscr{L}')] = \delta(\mathscr{L}, \mathscr{L}')\delta_{\kappa\kappa'},\\
&[a^{(\kappa)}(\mathscr{L}), a^{(\kappa')}(\mathscr{L}')] = 0,\\
&[a^{(\kappa)\dagger}(\mathscr{L}), a^{(\kappa')\dagger}(\mathscr{L}')] = 0,
\end{aligned} \tag{2.24}$$

where $\delta(\mathscr{L}, \mathscr{L}')$ is unity if all indices of $\mathscr{L}$ match those of $\mathscr{L}'$, and vanishes otherwise.

The realisation we will seek for a given Lie algebra will be a suitably chosen linear superposition of the bilinear forms $a^{(\kappa)\dagger}(\mathscr{L})a^{(\kappa)}(\mathscr{L}')$ for the operators $E_{\pm\alpha}$ and $\boldsymbol{H}$. The irreps will be realised by homogeneous polynomials (or monomials) of $a^{\dagger}(\mathscr{L})$'s operating on the 'vacuum state', $|0\rangle$.

Additional new notation will be introduced later, as needed.

3. Classical groups

3.1. *The groups D(n)*

We will carry out the analysis for these groups in some detail, since they present most of the features characteristic of the other classical groups. The latter will be treated subsequently, type by type, but in a more cursory fashion.

3.1.1. Realisations of the algebra. We will gradually build up a hierarchy of realisations of the algebra of D(n). At each stage, the algebra will be correctly realised. However, each succeeding stage will allow for the possibility of realising more and more types of irreps, until, at the last stage, it will be possible to realise all irreps of D(n).

To begin with, it is useful to quote the detailed form of (2.4) for this case, as given in FFMa (together with a specific solution for the phase factors, obtained in that reference):

$$[E_{(p-q)}, E_{(q-r)}] = d(p,q,r)E_{(p-r)}, \tag{3.1}$$

where

$$p, q, r = \pm1, \pm2, \ldots, \pm n, \qquad p\neq q\neq r, \tag{3.2}$$

$$d(p,q,r) = \varepsilon(p+q)\varepsilon(q+r)\varepsilon(r+p), \tag{3.3}$$

and

$$\begin{aligned}
&\varepsilon(x) = +1, \qquad x>0,\\
&\varepsilon(x) = -1, \qquad x<0.
\end{aligned} \tag{3.4}$$

We attempt to realise the algebra by letting

$$H=\sum_p \mu_p a^\dagger(p)a(p)\equiv\sum \mu_p N(p), \tag{3.5}$$

and

$$\begin{aligned} E_{(p-q)} &= \theta(p,q)\{a^\dagger(p)a(q)-a^\dagger(-q)a(-p)\} \\ &\equiv \theta(p,q)\{N(p,q)-N^\dagger(-p-q)\}, \qquad p\neq q, \end{aligned} \tag{3.6}$$

where $\theta(p,q)$ is a phase factor, the standard form of the number operator appears in (3.5) and other bilinear forms $a^\dagger a$ are defined as $N(p,q)$ in (3.6).

The algebra, defined by (2.1)-(2.3) and (3.1), and the additional conditions (2.8) and (2.17) yield conditions on $\theta(p,q)$. However, the problem is so trivial, because of the factorised form of the phase $d(p,q,r)$, (3.3), that we can immediately guess the solution. It is

$$\theta(p,q)=\varepsilon(p+q). \tag{3.7}$$

The algebra of D(n) is fully realised by the generators defined in (3.5) and (3.6) (with (3.7)). However, it will be apparent from the discussion of § 3.1.2 that only the elementary irrep† (which we will call f, for 'fundamental') associated with the 'unbranched' terminal point of the Dynkin diagram of D(n), and related irreps (built up from the symmetric parts of the Kronecker products of this elementary irrep) can be generated in this scheme. In other words, using our abbreviated version of the Dynkin-Patera notation, only the irreps

$$\begin{aligned} &[\mathscr{T}]=[l_1], \qquad l=1,2,\ldots, \\ &f\equiv[1_1] \end{aligned} \tag{3.8}$$

can be constructed with SHO operators which appear in (3.5) and (3.6).

The scheme can be generalised trivially to allow the realisation of all the basic irreps, with the exception of the two elementary spinor irreps, by introducing another index, κ. (This fact will be demonstrated in § 3.1.2.) We thus realise the algebra by setting

$$H=\sum_{p,\kappa}\mu_p a^{(\kappa)\dagger}(p)a^{(\kappa)}(p)\equiv\sum \mu_p N^{(\kappa)}(p), \tag{3.9}$$

and

$$E_{(p-q)}=\varepsilon(p+q)\textstyle\sum''_{(p,q)}\{N^{(\kappa)}(p,q)-N^{(\kappa)\dagger}(-p,-q)\}, \tag{3.10}$$

where

$$\kappa=1,2,\ldots,n-2. \tag{3.11}$$

The $N^{(\kappa)}$'s are obvious generalisations of N's defined in (3.5) and (3.6) and $\Sigma''_{(p,q)}$ means that the p and q indices are not summed over.

In other words, we can now realise all the basic irreps generated from the antisymmetric parts of the Kronecker products of $[1_1]$, or

$$[\mathscr{T}]=[1_\kappa]. \tag{3.12}$$

† This nomenclature is defined in Dynkin (1957), p 346, or Wybourne (1974), p 116. The symbol τ_1 is used for f in the latter. In Feldman *et al* (1981) and FFMa, we call f 'the quark', and use the symbol q for it, but since quarks can also be put into spinor irreps (Casalbuoni 1980, Casalbuoni and Gatto 1980), this earlier notation is not a felicitous one in the present context.

The last, and this time non-trivial, step in our hierarchy of realisations of the D(n) algebra is intended to allow for the realisation of the two elementary irreps associated with the two 'branched' end points. These are the elementary spinor irreps, which we will call $\sigma_{(o)}$, $\sigma_{(e)}$ (see Dynkin (1957, pp 350, 351) and Wybourne (1974, p 121) where σ_1, σ_2 instead of $\sigma_{(o)}$, $\sigma_{(e)}$ is used),

$$\sigma_{(o)} \equiv [1_{n-1}], \qquad \sigma_{(e)} \equiv [1_n]. \tag{3.13}$$

The subscript (o) ((e)) denotes an odd (even) number of minus signs in the weights of the representation.

We consider embedding the group† D(n) so that it is a non-regular subgroup of a group D(N):

$$D(N) \supset D(n), \qquad n \geqslant 4, \tag{3.14}$$

such that the fundamental irrep of D(N), $f_{D(N)}$, goes into the direct sum of the three terminal point irreps of D(n), i.e.

$$f_{D(N)} \to (f \oplus \sigma_{(o)} \oplus \sigma_{(e)})_{D(n)}. \tag{3.15}$$

From the dimensions of the various irreps involved, we obtain the condition

$$N = n + 2^{n-1}, \qquad n \geqslant 4. \tag{3.16}$$

We encounter no special problem for the first non-trivial case, $n = 4$, for which the Dynkin diagram is symmetric, and for which any of the terminal points can be associated with any one of f, $\sigma_{(e)}$ or $\sigma_{(o)}$.

The problem of obtaining the generators of a group, g, as superpositions of the generators of a group of higher rank, $\mathscr{G}$, where g is a non-regular subgroup of $\mathscr{G}$, was considered in general and solved for a large number of specific cases in FFMb. We will employ the methods of this reference as much as possible.

The root and weight space of D(N) can be defined by $2N$ orthonormal vectors

$$2n \qquad \boldsymbol{\lambda}_p, \qquad p = \pm 1, \pm 2, \ldots, \pm n, \tag{3.17}$$

$$\begin{gathered} 2(N-n) = 2^n \qquad \boldsymbol{\lambda}(p+q+r+\ldots), \\ p, q, r, \ldots = \pm 1, \pm 2, \ldots, \pm n, \qquad p \neq q \neq r \neq \ldots. \end{gathered} \tag{3.18}$$

There are n separate labels $p, q, r, \ldots$, (the sum total of index values is not relevant) and each distribution of plus or minus signs defines a unique $\boldsymbol{\lambda}$. The many-index $\boldsymbol{\lambda}$'s are symmetric in the index labels. This property is incorporated by using plus signs between them. The analogue of (2.9) holds and (see (2.11))

$$\begin{gathered} \boldsymbol{\lambda}_{-p} = -\boldsymbol{\lambda}_p, \\ \boldsymbol{\lambda}(-p-q-r\ldots) = -\boldsymbol{\lambda}(p+q+r\ldots). \end{gathered} \tag{3.19}$$

We choose the projections

$$\begin{gathered} \boldsymbol{\lambda}_p \to \boldsymbol{\mu}_p \equiv (p), \\ \boldsymbol{\lambda}(p+q+r\ldots) \to \tfrac{1}{2}(\boldsymbol{\mu}_p + \boldsymbol{\mu}_q + \boldsymbol{\mu}_r \ldots) \equiv \tfrac{1}{2}(p+q+r\ldots), \\ p, q, r, \ldots = \pm 1, \pm 2, \ldots, \pm n, \end{gathered} \tag{3.20}$$

† The results reduce to trivial ones in the cases $n = 2, 3$, for which there are no branched end points: for $n = 2$, $N = 2$ and $D(2) \simeq A(1) \otimes A(1)$; for $n = 3$, $N = 4$, $D(3) \simeq A(3)$ and $D(4) \supset B(3) \supset D(3)$, so that D(3) is a regular non-maximal subgroup of D(4).

where the $\boldsymbol{\mu}_p$ define the weight and root space of D(n). (The convenience of using summation signs for the independent labels in $\boldsymbol{\lambda}$ is now apparent.) From (3.20), the projection of the roots D(N) to the roots of D(n) is

$$\left.\begin{array}{l}\boldsymbol{\lambda}_p-\boldsymbol{\lambda}_q\\ \boldsymbol{\lambda}(p-q+r+s\ldots)+\boldsymbol{\lambda}(p-q-r-s\ldots)\end{array}\right\}\rightarrow\boldsymbol{\mu}_p-\boldsymbol{\mu}_q. \tag{3.21}$$

Thus,

$$1+\tfrac{1}{2}\cdot 2^{n-2}=1+2^{n-3} \tag{3.22}$$

roots of D(N) project into a single root of D(n). Each of the generators, $E_{(p-q)}$, of D(n) will be a linear superposition of the $1+2^{n-3}$ generators of D(N), corresponding to the D(N) roots listed in (3.21). A given component of $\boldsymbol{H}$ of D(n) will also be a linear superposition of $1+2^{n-1}$ components of the $\boldsymbol{\mathscr{H}}$ of D(N).

Indeed, we explicitly have

$$\boldsymbol{\mathscr{H}}=\sum\boldsymbol{\lambda}_p\mathscr{H}(p)+\sum\boldsymbol{\lambda}(p+q+r\ldots)\mathscr{H}(p+q+r\ldots), \tag{3.23}$$

where there are a total of $2n+2^n$ terms in the two sums.

If we define $\boldsymbol{H}$ as

$$\boldsymbol{H}=\sum_{p=-n}^{n}\boldsymbol{\mu}_pH(p)=\sum_{p=1}^{n}\boldsymbol{\mu}_p[H(p)-H(-p)], \tag{3.24}$$

we obtain from (3.20) ($\Sigma'_{(p)}$ below means that the index p is not summed over)

$$H(p)=\mathscr{H}(p)+\tfrac{1}{2}\sum\nolimits'_{(p)}\mathscr{H}(p+q+r\ldots). \tag{3.25}$$

$\boldsymbol{H}$ has n independent components, as it must.

So far, we have followed the derivation of FFMb step by step. It is at this point that the present approach diverges from that of FFMb, and we gain a considerable advantage over FFMb by using SHO operators. In order to determine the coefficients of the D(N) generators appearing in $E_{(p-q)}$, we must make use of the equations (2.1)-(2.3) and (3.1) which determine *both* the algebras of D(N) and of D(n). But, in order to make use of (3.1) for D(N), i.e. to evaluate $d(p, q, r)$'s which appear in it, we have to associate a single numerical index, a single 'address', with the n index symbol which is the argument of $\boldsymbol{\lambda}(p+q+r+\ldots)$. This is a daunting task for arbitrary n. On the other hand, if we think in terms of SHO realisations of the generators, we have no such problems, since the algebra of SHO's, equations (2.24), replaces the algebra of D(N) (in particular (3.1)) in our considerations. There is no phase in equation (2.24) and thus the 'address' problem is circumvented. We still have the phases of the superposition coefficients to obtain, but these are determined by the algebra of D(n), for which there is no 'address' problem to begin with. We therefore take, as the final version of the D(n) algebra realisation, the generators (3.24), with

$$\begin{aligned}H(p)&=\sum\nolimits'_{(p)}N^{(\kappa)}(p)+\frac{1}{2}\sum\nolimits'_{(p)}N(p+q+r\ldots),\\ N(p+q+r+\ldots)&\equiv a^{\dagger}(p+q+r+\ldots)a(p+q+r+\ldots),\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}E_{(p-q)}=\varepsilon(p+q)&\sum\nolimits''_{(p,q)}\{N^{(\kappa)}(p,q)-N^{(\kappa)\dagger}(-p,-q)\}\\ &+\sum\nolimits''_{(p,q)}\psi(p,q;r,s,\ldots)N(p-q;r+s+\ldots),\end{aligned} \tag{3.27}$$

where $\Sigma''_{(p,q)}$ is defined below (3.10) and

$$N(p-q;r+s+\ldots)\equiv a^{\dagger}(p-q+r+s+\ldots)a(-p+q+r+s+\ldots). \tag{3.28}$$

The other new symbol in (3.27), $\psi(p,q;r,s,\ldots)$, is a newly introduced phase factor. To save time, we have postulated a form for $E_{(p-q)}$ which already takes into account some features of the D(n) algebra: the fact that one phase is $\varepsilon(p+q)$ and that $|\psi|=1$. We impose the subsidiary conditions (2.8) and (2.17) on $E_{(p-q)}$, as given in (3.27). We then demand that $\boldsymbol{H}$, equation (3.26), and $E_{(p-q)}$, equation (3.27), satisfy the D(n) algebra, (2.1)-(2.3) and (3.1). These requirements result in the following conditions on the phases, ψ:

$$\psi(p,q;r,s,\ldots) \text{ is symmetric in } (p,q) \text{ and in } (r,s,\ldots), \tag{3.29}$$

$$\psi(p,q;r,s,\ldots)=\psi(-p,-q;r,s\ldots)=-\psi(p,q;-r,-s,\ldots), \tag{3.30}$$

and

$$d(p,q,r)=-\psi(p,q;r,s,\ldots)\psi(q,r;p,s,\ldots)\psi(r,p;q,s,\ldots), \tag{3.31}$$

as well as other equations which can be derived from (3.29)-(3.31) and therefore will not be listed. The equations (3.29)-(3.31) are algebraically identical to the conditions on the structure constant phases derived for the algebras of the groups E(8), E(7), E(6) and F(4) in† FFMa. We do possess explicit solutions in this reference of equations (3.29)-(3.31) for the phases $\psi_{(e)}$ for $n=4$, 6, 8 and $\psi_{(o)}$ for $n=4$, 5 (as well as the phase χ for B(4), which is defined below). Since F(4) ⊃ B(4) ⊃ D(4), E(6) ⊃ D(5) ⊕ U(1), E(7) ⊃ D(6) ⊕ A(1) and E(8) ⊃ D(8), these phases appear in the algebras of the larger groups. It is not difficult to generalise these solutions to arbitrary n, ($n\geq 4$).

The results are,

n odd:

$$\psi(p,q;r,s,t\ldots)_{(e)}=-\textstyle\prod''_{(p,q)}\varepsilon(|p|-|l_-|)\varepsilon(|q|-|l_-|), \tag{3.32}$$

$$\psi(p,q;r,s,t\ldots)_{(o)}=+\textstyle\prod''_{(p,q)}\varepsilon(|p|-(l_+|)\varepsilon(|q|-|l_+|), \qquad p,q\neq l_+,l_-. \tag{3.33}$$

The subscripts (e) ((o)) indicate an even (odd) number of minus signs in all of the indices of ψ. The symbols $l_+(l_-)$ are the values of the positive (negative) indices, and $\Pi''_{(p,q)}$ indicates that p and q are excluded from the product over the l_- or l_+ indices.

n even:

$$\psi(p,q;r,s,t,\ldots,-n)=-\textstyle\prod'''_{(p,q,-n)}\varepsilon(|p|+l)\varepsilon(|q|+l), \tag{3.34}$$

$$\psi(p,q;r,s,t,\ldots,+n)=+\textstyle\prod'''_{(p,q,+n)}\varepsilon(|p|-l)\varepsilon(|q|-l), \tag{3.35}$$

$$p,q=\pm1,\pm2,\ldots,\pm(n-1); \qquad p,q\neq l$$

with $\Pi'''_{(p,q,-n)}$ an obvious generalisation of $\Pi''_{(p,q)}$ above.

The list of phases continues with

$$\psi(p,-n;r,s,t\ldots)=\prod_{|l|<|m|}{}''_{(p,-n)}\varepsilon(l+m), \qquad p,r,s>0, \tag{3.36}$$

$$\psi(p,n;r,s,t\ldots)=(-1)^{p+1}\psi(p,-n;r,s,t\ldots), \tag{3.37}$$

† See FFMa, equations (B.20) and (B.21) for E(8) and E(7), E(6) and (B.24) and (B.26) for F(4).

where r, s, t are numerically ordered:

$$r < s < |t| < \ldots,$$

and (3.38)

$$p, r, s, |t|, \ldots = 1, 2, \ldots, n-1.$$

The remaining phases are fixed by the condition

$$\psi(p, -n; r, -s, -t, -u, \ldots) \equiv -\psi(p, -n; r, s, t, u, \ldots), \qquad p, r, s > 0, \tag{3.39}$$

and one of (3.30),

$$\psi(p, -n; -r, -s, -t, \ldots) = -\psi(p, -n; r, s, t, \ldots), \qquad p, r > 0. \tag{3.30}$$

We may note that for

$$n = 4l, \qquad l = 1, 2, 3, \ldots, \tag{3.40}$$

the numerical ordering, (3.38), and the requirements $r > 0$, $s > 0$ and (3.39) are not necessary in the phase definitions (3.36) and (3.37): (3.30), (3.36)-(3.39) will in any case be automatically satisfied. However, for n values

$$n = 4l + 2, \qquad l = 1, 2, 3, \ldots, \tag{3.41}$$

this is not the case. The conditions as stated in (3.30), (3.36)-(3.39) cover both of these situations.

We do not wish to stress the tedious details of the expressions (3.32)-(3.39). Suffice it to say that we have demonstrated the existence of solutions, with real phase factors, to (3.29)-(3.31), for arbitrary n. Equation (3.31) requires the discovery of a factorised form, other than (3.3), for the *same* phase factor $d(p, q, r)$ as appears in (3.3).

3.1.2. Realisations of the irreps. We note the commutation relations

$$[\boldsymbol{H}, a^{(\kappa)\dagger}(p)] = \boldsymbol{\mu}_p a^{(\kappa)\dagger}(p), \tag{3.42}$$

$$[\boldsymbol{H}, a^{\dagger}(p+q+r+\ldots)] = \tfrac{1}{2}(\boldsymbol{\mu}_p + \boldsymbol{\mu}_q + \boldsymbol{\mu}_r + \ldots) a^{\dagger}(p+q+r+\ldots), \tag{3.43}$$

$$[\boldsymbol{H}, a^{(\kappa)\dagger}(p) a^{(\lambda)\dagger}(q)] = (\boldsymbol{\mu}_p + \boldsymbol{\mu}_q) a^{(\kappa)\dagger}(p) a^{(\lambda)\dagger}(q), \tag{3.44}$$

$$\vdots \qquad\qquad \vdots$$

$$[E_{(p-q)}, a^{(\kappa)\dagger}(q)] = a^{(\kappa)\dagger}(p). \tag{3.45}$$

$$\vdots \qquad\qquad \vdots$$

They follow from the commutation relations (2.24), together with the definition of the components of $\boldsymbol{H}$, equation (3.26), and of $E_{(p-q)}$, equation (3.27). They, and similar equations, are required in checking that the states we are about to define satisfy the equations of (2.19) and (2.20).

We now have the following irreps:

Scalar irrep

$$|\mathbf{0}, [0]\rangle \equiv |0\rangle, \tag{2.21}$$

defined by

$$a(\mathscr{L})|0\rangle = 0, \qquad \forall \mathscr{L}. \tag{3.46}$$

The three elementary irreps. They are given by

$$f\colon |\boldsymbol{\mu}_p, [1_1]\rangle = a^{(\kappa)\dagger}(p)|0\rangle, \tag{3.47}$$

$$\sigma_{(\mathrm{o})}\colon \left|\left(\tfrac{1}{2}\sum_{(\mathrm{o})}(\pm)\boldsymbol{\mu}_p\right), [1_{n-1}]\right\rangle = a^{\dagger}_{(\mathrm{o})}(p+q+r+\ldots)|0\rangle, \tag{3.48}$$

$$\sigma_{(\mathrm{e})}\colon \left|\left(\tfrac{1}{2}\sum_{(\mathrm{e})}(\pm)\boldsymbol{\mu}_p\right), [1_n]\right\rangle = a^{\dagger}_{(\mathrm{e})}(p+q+r+\ldots)|0\rangle. \tag{3.49}$$

See below (3.13) for the definition of (o) and (e).

The irrep f is defined redundantly, for any fixed value of κ. The $(n-2)$ fold redundancy is trivial: a $2n$-dimensional vector $\boldsymbol{r}^{(1)}$ is as good a vector as a $2n$-dimensional vector $\boldsymbol{r}^{(2)}$.

The remaining basic irreps. The adjoint, a, is given by

$$a\colon |(\boldsymbol{\mu}_p - \boldsymbol{\mu}_q), [1_2]\rangle = 2^{-1/2}\,\mathrm{det}_{(\kappa_1,\kappa_2)}\, a^{(\kappa_1)\dagger}(p)a^{(\kappa_2)\dagger}(-q)|0\rangle, \tag{3.50}$$

where

$$\mathrm{det}_{(\kappa_1,\kappa_2)}\, a^{(\kappa_1)\dagger}(p)a^{(\kappa_2)\dagger}(-q) = a^{(\kappa_1)\dagger}(p)a^{(\kappa_2)\dagger}(-q) - (\kappa_1 \leftrightarrow \kappa_2). \tag{3.51}$$

As in the case of f, there is a redundancy in the definition of the adjoint. There are $\frac{1}{2}(n-2)(n-3)$ different realisations of it, corresponding to the choice of different (κ_1, κ_2) pairs.

The irrep $[1_3]$ will correspond to a $\mathrm{det}_{(\kappa_1,\kappa_2,\kappa_3)}$ over three $a^{\dagger}$'s, and so on. The redundancy will disappear for $[1_{n-2}]$. The number of κ indices were chosen so that this last basic irrep could be constructed.

All other irreps. All the other irreps can be constructed from Kronecker products of the basic irreps. We follow a slightly different route. We construct outer products of different $a^{\dagger}$ combinations, rather than states, obtaining irreducible subsets of them in a manner completely analogous to the reduction of Kronecker products for states. We then allow the resulting homogeneous polynomial in $a^{\dagger}$'s to operate on the vacuum state.

This procedure will give rise to further redundant irreps. For example, the state

$$n^{-1/2}\textstyle\sum'_{(\kappa)} a^{(\kappa)\dagger}(p)a^{(\kappa)\dagger}(-p)|0\rangle \tag{3.52}$$

is a possible realisation of the scalar irrep. This fact can easily be checked by applying $\boldsymbol{H}$ and all the step-up and step-down operators $E_{(p-q)}$ to this state.

This type of redundancy is not as trivial as that previously discussed, which was related to the (κ) superscripts. In order to eliminate it, we will append the rule that the irreps must be realised in terms of the minimal number of $a^{\dagger}$ factors operating on the vacuum and the minimal number of (κ) indices. If more complicated forms arise, say in the reduction of a Kronecker product, they must be replaced by the simpler forms. Thus, for the state considered in (3.52), we must make the replacement

$$n^{-1/2}\textstyle\sum'_{(\kappa)} a^{(\kappa)\dagger}(p)a^{(\kappa)\dagger}(-p)|0\rangle \to |0\rangle. \tag{3.53}$$

We will call this process 'SHO reduction'.

A typical non-basic irrep which is an intrinsically two particle state and cannot be SHO reduced further is

$$|(\boldsymbol{\mu}_p-\boldsymbol{\mu}_q),[2_1]\rangle=\left(1+\delta_{p,-q}-\frac{1}{n}\delta_{p,q}\right)^{-1/2}$$

$$\times\left(a^{(\kappa)\dagger}(p)a^{(\kappa)\dagger}(-q)-\frac{\delta_{p,q}}{n}\sum\nolimits'_{(\kappa)}a^{(\kappa)\dagger}(l)a^{(\kappa)\dagger}(-l)\right)|0\rangle. \tag{3.54}$$

The leading factor in (3.54) is a normalisation factor which differs from unity for elements of weights $2\boldsymbol{\mu}_p$ and $\mathbf{0}$. The latter elements are defined symmetrically, but redundantly, so that

$$\sum_p|\mathbf{0}_p,[2_1]\rangle=0. \tag{3.55}$$

Thus, there are $n-1$ independent elements with weights $\mathbf{0}$. Since there are, in addition, $\binom{n}{2}\cdot 2^2=2n(n-1)$ elements with weights $\boldsymbol{\mu}_p-\boldsymbol{\mu}_q$, $|p|\neq|q|$, and $2n$ elements with weights $2\boldsymbol{\mu}_p$, there are a total of

$$\dim[2_1]_{\mathrm{D}(n)}=n(2n+1)-1 \tag{3.56}$$

elements in this irrep, the desired result.

We also note that there is only one independent superscript κ in (3.54), rather than two, κ_1 and κ_2. We could have used two superscripts, but would then have had to specify a reduction of the products, and kept only the symmetric (κ_1,κ_2) combinations. We will *always* take equal κ limits for such symmetric combinations, using our previously stated SHO reduction rules.

We have thus exhibited, or indicated a method of constructing, each of the irreps at least once.

3.2. *The groups B(n)*

3.2.1. Realisation of the algebra. Since the elementary irrep f has $n-2$ antisymmetrised basic irreps associated with it, rather than $n-3$, as in the case of $\mathrm{D}(n)$, (3.11) is altered to

$$\kappa=1,2,\ldots,n-1. \tag{3.57}$$

In addition to the $\mathrm{D}(n)$ generators, there is a new set of generators, $E_{(p)}$. Additional sets of commutators of type (2.4) exist (see FFMa, equations (3.9) and (3.10), with a specific set of phase solutions, (3.11)). $\boldsymbol{H}$ and $H(p)$ have the same forms as in $\mathrm{D}(n)$ (see (3.24) and (3.25)), as do the generators $E_{(p-q)}$ (see (3.27)).

The new generators are given by

$$E_{(p)}=\sum\nolimits'_{(p)}\{N^{(\kappa)}(p,0)+N^{(\kappa)\dagger}(-p,0)+\chi(p;q,r\ldots)N(p;q+r\ldots)\}, \tag{3.58}$$

where

$$N(p;q+r+\ldots)\equiv a^{\dagger}(p+q+r+\ldots)a(-p+q+r+\ldots), \tag{3.59}$$

and where $\chi(p;q,r\ldots)$ are new phase factors, and we have also introduced another set of $n-1$ Bose oscillators $[a^{(\kappa)\dagger}(0)$ and $a^{(\kappa)}(0)]$, associated with zero weight vectors. Different labels are once again taken to have different values. The commutators of the algebra and the subsidiary conditions (2.8) and (2.17) yield the new phase conditions

$$\chi(p;q,r\ldots)\qquad \text{is symmetric in } (q,r,\ldots), \tag{3.60}$$

$$\chi(p;q,r\ldots)=\chi(-p;q,r\ldots)=-\chi(p;-q,-r\ldots), \tag{3.61}$$

$$\psi(p,q;r,s\ldots)=-\varepsilon(p+q)\chi(p;q,r,s\ldots)\chi(q;p,r,s\ldots). \tag{3.62}$$

A set of solutions for the new phases is
n odd:

$$\chi(p;q,r\ldots)_{(\mathrm{e})}=-\textstyle\prod'_{(p)}\varepsilon(|p|-|l_-|), \tag{3.63}$$

$$\chi(p;q,r\ldots)_{(\mathrm{o})}=\textstyle\prod'_{(p)}\varepsilon(|p|-|l_+|),\qquad p\neq l_+,l_-. \tag{3.64}$$

$\Pi'_{(p)}$ is defined analogously to $\Pi''_{(p,q)}$, below (3.33). For the definitions of l_+ and l_-, see the text following (3.33). Of course in the present case, because of (3.61), (e) and (o) refer to the indices other than p.
n even:

$$\chi(p;q,r,\ldots,-n)=-\textstyle\prod''_{(p,-n)}\varepsilon(|p|+l), \tag{3.65}$$

$$\chi(p;q,r,\ldots,+n)=\textstyle\prod''_{(p,n)}\varepsilon(|p|-l), \tag{3.66}$$

$$p=\pm1,\pm2,\ldots,\pm(n-1);\ p\neq l.$$

The remaining phases are

$$\begin{aligned}&\chi(\pm n;p,q,r,s\ldots)=-\prod_{|l|<|m|}{}'_{(\pm n)}\varepsilon(l+m),\qquad p,q,r>0,\\ &p,q,r,|s|,\ldots=1,2,\ldots,n-1;\qquad p<q<r<|s|<\ldots,\end{aligned} \tag{3.67}$$

with

$$\begin{aligned}&\chi(\pm n;p,q,-r,-s\ldots)\equiv-\chi(\pm n;p,q,r,s\ldots), && p,q,r>0,\\ &\chi(\pm n;p,-q,-r\ldots)\equiv-\chi(\pm n;p,q,r\ldots), && p,q>0,\\ &\chi(\pm n;-p,-q,-r\ldots)\equiv-\chi(\pm n;p,q,r\ldots), && p>0.\end{aligned} \tag{3.68}$$

Note that, in analogy with (3.36) and (3.37), $p, q, r,\ldots$ are taken in numerical order in (3.67) and (3.68). As in the case of $\mathrm{D}(n)$, these requirements, (3.68) and the positivity conditions on the indices p, q, r are only needed for the case of $n=6, 10, 14,\ldots$. For the case of $n=4, 8, 12,\ldots$ (3.68) is automatically satisfied, even if these conditions are not imposed.

The χ's are obtained from the ψ's essentially by factorising the latter (see (3.62)).

3.2.2. Realisations of the irreps. The changes from the $\mathrm{D}(n)$ case are trivial. We note

$$[\boldsymbol{H}, a^{(\kappa)\dagger}(0)]=0. \tag{3.69}$$

Scalar irrep. Equations (2.21) and (3.46) hold. Note that the set $\mathscr{L}$ includes 0, and

$$\kappa=1,2,\ldots,n-1,\qquad p=\pm1,\pm2,\ldots,\pm n. \tag{3.70}$$

The two elementary irreps. f: (3.47), and an additional element

$$|\mathbf{0},[1_1]\rangle=a^{(\kappa)\dagger}(0)|0\rangle, \tag{3.71}$$

$$\sigma\colon \sigma=\sigma_{(\mathrm{e})}\oplus\sigma_{(\mathrm{o})}, \tag{3.72}$$

where $\sigma_{(e)}$ and $\sigma_{(o)}$ are given in (3.48) and (3.49). These results were to be expected, since $\mathrm{B}(n) \supset \mathrm{D}(n)$.

The other irreps. They are identical to those defined in connection with $\mathrm{D}(n)$, provided the set of labels is extended, as indicated in (3.70).

3.3. *The groups C(n)*

3.3.1. Realisations of the algebra. This is the simplest of all classical groups to treat. There are no spinors, and all irreps can be constructed from f, with the other basic irreps being a total of $n-1$ associated antisymmetrised irreps (with suitable vanishing trace conditions). Thus we require

$$\kappa = 1, 2, \ldots, n. \tag{3.73}$$

The algebra in this case leads to phase solutions which are entirely different from those of $\mathrm{D}(n)$ and $\mathrm{B}(n)$. (For the commutation relations in which the phases are defined, see FFMa, equations (3.12)-(3.14), and for specific solutions, equations (3.15) and (3.16). Note the error in (3.16): $\bar{\sigma}(p, q)$ should read $\bar{\sigma}(q, p)$.) The generators $\boldsymbol{H}$ are given by (3.9) (noting (3.73)). The structure of the $E_{(p-q)}$ generators is the same as in (3.10). They, and the generators $E_{(2p)}$, are given in detail by

$$\begin{aligned} E_{(p-q)} &= \tfrac{1}{2}(1+\varepsilon(p)-\varepsilon(q)+\varepsilon(p)\varepsilon(q)) \\ &\quad \times \textstyle\sum''_{(p,q)} \{\varepsilon(p) N^{(\kappa)}(p, q) - \varepsilon(q) N^{(\kappa)+}(-p, -q)\}, \end{aligned} \tag{3.74}$$

$$E_{(2p)} = \sqrt{2}\textstyle\sum'_{(p)} N^{(\kappa)}(p, -p), \qquad p = \pm 1, \pm 2, \ldots, \pm n. \tag{3.75}$$

3.3.2. Realisations of the irreps

Scalar and elementary irreps. The scalar and f irreps are those given for $\mathrm{D}(n)$ (but note (3.73)).

The remaining basic irreps. The only minor difficulty for $\mathrm{C}(n)$ arises here, since the antisymmetric parts of Kronecker products are reducible. We illustrate for two irreps, which make the general pattern clear. The irrep constructed from the antisymmetric product of two a^{+}'s is

$$|(\boldsymbol{\mu}_p + \boldsymbol{\mu}_q), [1_2]\rangle = \frac{1}{\sqrt{2}} \det_{(\kappa_1,\kappa_2)} a^{(\kappa_1)+}(p) a^{(\kappa_2)+}(q)|0\rangle, \qquad |p| \neq |q|, \tag{3.76a}$$

$$\begin{aligned} |\mathbf{0}_p, [1_2]\rangle &= \frac{1}{\sqrt{2}}\left(\frac{n}{n-1}\right)^{1/2} \\ &\quad \times \det_{(\kappa_1,\kappa_2)}\left[a^{(\kappa_1)+}(p) a^{(\kappa_2)+}(-p) - \frac{1}{n}\textstyle\sum''_{(\kappa_1,\kappa_2)} a^{(\kappa_1)+}(l) a^{(\kappa_2)+}(-l)\right]|0\rangle. \end{aligned} \tag{3.76b}$$

The zero weight elements are defined redundantly, with the single condition

$$\sum_p |\mathbf{0}_p, [1_2]\rangle = 0, \tag{3.76c}$$

so that, since $p=q$ is excluded, we get the expected number of elements for the irrep:

$$\dim[1_2]_{C(n)}=\binom{n}{2}2^2+n-1=2n^2-n-1. \tag{3.77}$$

The irrep constructed from the antisymmetric product of three $a^{\dagger}$'s is

$$|(\boldsymbol{\mu}_p+\boldsymbol{\mu}_q+\boldsymbol{\mu}_r),[1_3]\rangle=\frac{1}{\sqrt{3!}}\det_{(\kappa_1,\kappa_2,\kappa_3)} a^{(\kappa_1)+}(p)a^{(\kappa_2)+}(q)a^{(\kappa_3)+}(r)|0\rangle, \tag{3.78a}$$

$$\begin{aligned}|(\boldsymbol{\mu}_p+\mathbf{0}_q),[1_3]\rangle=&\left(\frac{n-1}{n-2}\right)^{1/2}\frac{1}{\sqrt{3!}}\det_{(\kappa_1,\kappa_2,\kappa_3)} a^{(\kappa_1)+}(p)\\&\times\left[a^{(\kappa_2)+}(q)a^{(\kappa_3)+}(-q)-\frac{1}{n-1}\sum\nolimits'''_{(\kappa_2,\kappa_3,p)} a^{(\kappa_2)+}(l)a^{(\kappa_3)+}(-l)\right]|0\rangle,\\&|p|\neq|q|.\end{aligned} \tag{3.78b}$$

The $\boldsymbol{\mu}_p$ weight elements are defined redundantly, with one condition for each p,

$$\sum_q|(\boldsymbol{\mu}_p+\mathbf{0}_q),[1_3]\rangle=0, \qquad |p|\neq|q|, \tag{3.78c}$$

or $2n$ conditions in all. We again get the expected number of elements for this irrep:

$$\dim[1_3]_{C(n)}=\binom{n}{3}\cdot 2^3+2n(n-2)=\tfrac{2}{3}n(n-2)(2n+1). \tag{3.79}$$

The method for constructing all other basic irreps is now apparent.

The adjoint irrep. We illustrate only this single non-basic irrep. It is

$$|(\boldsymbol{\mu}_p-\boldsymbol{\mu}_q),[2_1]\rangle=(1+\delta_{p,-q})^{-1/2}a^{(\kappa)+}(p)a^{(\kappa)+}(-q)|0\rangle, \tag{3.80}$$

with $\binom{n}{2}\cdot 2^2$ elements $|p|\neq|q|$, $2n$ elements $p=-q$ and n zero weight elements, or, as expected,

$$\dim[2_1]_{C(n)}=n(2n+1). \tag{3.81}$$

3.4. *The groups A(n)*

The added complication here is that the basis space of $A(n)$ is defined in terms of $n+1$ orthonormal basis vectors $\boldsymbol{\mu}_p$. There are again no spinor irreps and all irreps can be constructed from a single f, with weights $\boldsymbol{\chi}_p$, where $\boldsymbol{\chi}_p$ is defined in (2.13). However, we prefer to discuss this case using two elementary irreps, f with weights $\boldsymbol{\chi}_p$, and $\bar{f}$ with weights $-\boldsymbol{\chi}_p$. The treatment of the adjoint irrep as well as of G(2) generators is more natural in such an approach than in one based on the construction of the irreps from a single f.

3.4.1. Realisation of the algebra. The generators in $\boldsymbol{H}$, where

$$\boldsymbol{H}=\sum\boldsymbol{\chi}_pH(p), \tag{3.82}$$

are

$$H(\pm|p|)=\sum\nolimits'_{(p)}N^{(\kappa)}(\pm|p|)-\frac{1}{n+1}\sum N^{(\kappa)}(\pm|q|), \tag{3.83}$$

so that $\boldsymbol{H}$ has the expected n, rather than $n+1$, independent components.

The step up or step down generators are

$$E_{(p-q)}=\sum''_{(p,q)}[N^{(\kappa)}(p,q)-N^{(\kappa)\dagger}(-p,-q)], \tag{3.84}$$

with

$$\begin{aligned} &p, q=1,2,\ldots,n+1, \text{or } -1,-2,\ldots,-(n+1),\\ &\kappa=1,2,\ldots,[n/2],\\ &\left[\frac{n}{2}\right]\equiv\begin{cases}\frac{1}{2}n, & n \text{ even},\\ \frac{1}{2}(n+1), & n \text{ odd}.\end{cases}\end{aligned} \tag{3.85}$$

3.4.2. Realisations of the irreps. We note that (3.42)-(3.44) are modified by the replacement

$$\boldsymbol{\mu}_p\to\boldsymbol{\chi}_p\colon [\boldsymbol{H}, a^{(\kappa)\dagger}(p)]=\boldsymbol{\chi}_p a^{(\kappa)\dagger}(p), \qquad \text{etc.} \tag{3.86}$$

Scalar irrep. This is defined as for D(n), but with index ranges as given in (3.85).

The two elementary irreps. We have

$$f\colon |\boldsymbol{\chi}_p,[1_1]\rangle=a^{(\kappa)\dagger}(p)|0\rangle, \tag{3.87}$$

$$\bar{f}\colon |-\boldsymbol{\chi}_p,[1_n]\rangle=a^{(\kappa)\dagger}(-p)|0\rangle, \qquad p=1,2,\ldots,n+1. \tag{3.88}$$

The other basic irreps. They are obtained by taking the antisymmetric parts of the Kronecker products of $a^{(\kappa)\dagger}(p)$'s or $a^{(\kappa)\dagger}(-p)$'s. For n odd, both $a^{(\kappa)\dagger}(p)$ and $a^{(\kappa)\dagger}(-p)$ products lead to the same irrep for $[1_{(n+1)/2}]$.

The adjoint irrep. All other irreps can now be constructed. However, because of the special role it plays, we once more specifically give the adjoint irrep. It is

$$|(\boldsymbol{\chi}_p-\boldsymbol{\chi}_q),[1_1,1_n]\rangle\equiv|(\boldsymbol{\mu}_p-\boldsymbol{\mu}_q),[1_1,1_n]\rangle=a^{(\kappa)\dagger}(p)a^{(\kappa)\dagger}(-q)|0\rangle, \qquad p\neq q, \tag{3.89a}$$

$$\begin{aligned}&|\mathbf{0}_p,[1_1,1_n]\rangle=\left(\frac{n+1}{n}\right)^{1/2}\left[a^{(\kappa)\dagger}(p)a^{(\kappa)\dagger}(-p)-\frac{1}{n+1}\sum'_{(\kappa)}a^{(\kappa)\dagger}(l)a^{(\kappa)\dagger}(-l)\right]|0\rangle,\\ &\qquad p,q,l=1,2,\ldots,n+1,\end{aligned} \tag{3.89b}$$

with the redundancy

$$\sum_{p=1}^{n+1}|\mathbf{0}_p,[1_1,1_n]\rangle=0. \tag{3.89c}$$

4. Exceptional groups

Our emphasis in this section will be on the realisation of the algebras. The treatment of the irreps will be much sketchier than for the classical groups. We will give explicit expressions at most for the elementary irreps, often not even for all of these. However, for every exceptional group, the irreps given will be sufficient to generate all other irreps through Kronecker products.

The strategy we follow is the same as that which led to the realisation of the D(n) and B(n) algebras, yielding spinor irreps in terms of Bose oscillators: we embed each exceptional group of rank n, g_n, in an orthogonal group $\mathcal{G}_N$ of higher rank, such that N is as small as possible and g_n is a non-regular subgroup of $\mathcal{G}_N$. It must be emphasised in this connection that, in obtaining the phases which appear, those algebraic conditions in (2.4) for which the structure constants vanish can play an important role. In general they do not lead to additional independent constraints on the phases. However, in the case of F(4) and E(8), where the lowest dimensional elementary irreps contain more than one vanishing weight, such relations play a crucial role and lead to complex phase factors. These relations are spelled out in detail in FFMb, equations (4.52) and (4.53), and figures 1(a) and 1(b). They are applied to specific cases in appendix A of FFMb.

We consider each of the exceptional groups in order of increasing rank, and emphasise results, rather than the details of the ways in which these results have been obtained. We do this both for the sake of brevity, and because the approach we follow has been adequately illustrated in FFMb and in our present § 3.1.

4.1. G(2)

4.1.1. Realisation of the algebra. We note that

$$\mathrm{B}(3) \supset \mathrm{G}(2), \tag{4.1}$$

and that the relation

$$\begin{array}{ccc} [1_1] & \rightarrow & [1_2] \\ f & \rightarrow & f_1 \\ 7 & & 7 \end{array} \tag{4.2}$$

holds, where f_1 is one of the two elementary irreps of G(2). The other is f_2, where

$$f_2 \equiv a \equiv [1_1], \qquad \dim f_2 = 14. \tag{4.3}$$

The LHS of the relation (4.2), and similar relations for the other exceptional groups, refers to the group of higher rank, $\mathcal{G}_N$, and the RHS to the group of lower rank g_n($\mathcal{G}_N \supset g_n$, non-regular). The first row gives our version of the Dynkin–Patera indices; the second row repeats this information in terms of the symbols for the elementary irreps; the third row gives the dimension of the irrep listed above it.

We now project from the $\boldsymbol{\lambda}$ weight space of B(3) to the $\boldsymbol{\mu}$ weight space of G(2), analogously to the method followed in § 3.1:

$$\begin{aligned} &\mathbf{0} \rightarrow \mathbf{0} \\ &\boldsymbol{\lambda}_p \rightarrow \boldsymbol{\chi}_p \equiv \boldsymbol{\mu}_p - \frac{1}{3}\sum_{q=1}^{3} \boldsymbol{\mu}_q, \qquad p = \pm 1, \pm 2, \pm 3. \end{aligned} \tag{4.4}$$

Because of (2.14), the actual weight space of G(2) is two-, and not three-dimensional.

The resulting generators, with the appropriate phase solutions, are

$$\boldsymbol{H} = \sum \boldsymbol{\chi}_p H(p), \tag{4.5}$$

with

$$H(p) = \textstyle\sum'_{(p)} N^{(\kappa)}(p) - \frac{1}{3}\sum N^{(\kappa)}(q), \tag{4.6}$$

and

$$E_{(p-q)} = \varepsilon(p+q) \textstyle\sum''_{(p,q)} [N^{(\kappa)}(p,q) - N^{(\kappa)\dagger}(-p,-q)], \tag{4.7}$$

$$E_{\chi_p} = \frac{1}{\sqrt{3}} \sum_{\kappa=1}^{2} [\sqrt{2}(N^{(\kappa)}(p,0) + N^{(\kappa)\dagger}(-p,0))$$
$$+ (-1)^{p+1}\varepsilon(-q+r)(N^{(\kappa)}(-q,r) - N^{(\kappa)\dagger}(q,-r))], \tag{4.8}$$

with p, q, r in (4.7) and (4.8) all positive or all negative, $p \neq q \neq r$ and p, q, r in (4.8) cyclic.

As indicated in some of the explicitly written sums above, κ has the values

$$\kappa = 1, 2. \tag{4.9}$$

4.1.2. Realisation of the irreps. The scalar and $[1_2] \equiv f_1$ irreps have the, by now, familiar forms, in analogy with the classical groups. The adjoint irrep is generated from the Kronecker product of two $a^\dagger$'s

$$\begin{aligned} &[1_2]\otimes[1_2] = [0]\oplus[1_2]\oplus[1_1]\oplus[2_2] \\ &f_1\otimes f_1 \quad = s \quad \oplus f_1 \quad \oplus f_2 \quad \oplus S(2f_1) \\ &49 \qquad\quad = 1 \quad +7 \quad +14 \quad +27. \end{aligned} \tag{4.10}$$

The symbols introduced here are $s \equiv [0]$, for the scalar irrep, and $S(2f_1)$ for the irrep constructed from the symmetric product of two f_1's. The notation $A(2f)$ and $(f_1 f_2)$, to appear subsequently, denotes irreps constructed from antisymmetric products of two f's, and from products of f_1 and f_2, respectively. Using (4.10), we have for the irrep f_2:

$$|(\chi_p - \chi_q), [1_1]\rangle = (\pm)\frac{1}{\sqrt{2}} \det_{(1,2)} a^{(1)\dagger}(p) a^{(2)\dagger}(-q)|0\rangle, \qquad p, q = (\pm)\ (1, 2, 3), \tag{4.11a}$$

$$|\chi_p, [1_1]\rangle = \frac{1}{\sqrt{6}} \det_{(1,2)}[\sqrt{2} a^{(1)\dagger}(p) a^{(2)\dagger}(0)$$
$$+ (-1)^{p+1}\varepsilon(-q+r) a^{(1)\dagger}(-q) a^{(2)\dagger}(-r)]|0\rangle, \tag{4.11b}$$

$$|0_p, [1_1]\rangle = \frac{\sqrt{3}}{2} \det_{(1,2)}\left[a^{(1)\dagger}(p) a^{(2)\dagger}(-p) - \frac{1}{3}\sum_{q=1}^{3} a^{(1)\dagger}(q) a^{(2)\dagger}(-q)\right]|0\rangle. \tag{4.11c}$$

We call attention to the appearance of relative phase factors in (4.11b). The elements of vanishing weight are, as in (3.89b), defined redundantly (see (3.89c)). There are only two independent ones.

All other irreps can be constructed from f_1 by Kronecker products.

4.2. F(4)

From this point on, all the exceptional groups will contain roots with spinor weights. The trick of associating Bose oscillators with such weights, whether in irreps, such as those of D(n) and B(n), or in the generators themselves, is the same one: one must find a group D(N) in which the exceptional group can be non-regularly embedded.

In the present case, we have

$$\mathrm{D}(13) \supset \mathrm{F}(4), \tag{4.12}$$

and the relation

$$\begin{array}{ll} [1_1] \to [1_4] \\ f \quad \to f_1 \\ 26 \qquad 26. \end{array} \tag{4.13}$$

The other elementary irrep of F(4) is f_2,

$$f_2 \equiv a \equiv [1_1]. \tag{4.14}$$

In what follows we will choose index labels $p, q, r, s = \pm 1, \pm 2, \pm 3, \pm 4$. Label $x = \pm 13$ is such that the $\boldsymbol{\lambda}$ space to $\boldsymbol{\mu}$ space projection of $\boldsymbol{\lambda}_{13}$ is $\boldsymbol{\lambda}_{13} \to 0$.

The F(4) generators in terms of SHO operators are as follows.

The generators in $\boldsymbol{H}$, where

$$\boldsymbol{H} = \sum \boldsymbol{\mu}_p H(p), \tag{4.15}$$

are

$$H(p) = \sum\nolimits'_{(p)} N^{(\kappa)}(p) + \frac{1}{2} \sum\nolimits'_{(p)} N^{(\kappa)}(p+q+r+s). \tag{4.16}$$

Note that, while previously multi-index symbols such as $N^{(\kappa)}(p+q+r+s)$, defined in (3.26), did not carry the additional superscript index (κ), here, and from here on, they will. The corresponding SHO operators will also carry this (κ) superscript. The superscript is necessary for these terms in all of the exceptional groups, since otherwise the algebra is not closed.

The factor $\frac{1}{2}$ in (4.16) is traceable to the appearance of spinor weights among the F(4) roots.

The generators corresponding to roots of non-vanishing weight, with phase solutions explicitly given, are

$$\begin{aligned} E_{(p-q)} = \varepsilon(p+q) \sum\nolimits''_{(p,q)} [N^{(\kappa)}(p, q) - N^{(\kappa)\dagger}(-p, -q)] \\ + \sum\nolimits''_{(p,q)} \psi(p, q; r, s) N^{(\kappa)}(p-q; r+s), \end{aligned} \tag{4.17}$$

with $N^{(\kappa)}(p-q; r+s)$ defined in (3.28),

$$\begin{aligned} E_{(p)} = \frac{1}{\sqrt{2}} \Big(\sum\nolimits'_{(p)} [N^{(\kappa)}(p, x) + N^{(\kappa)\dagger}(-p, -x)] \\ + \sum\nolimits'_{(p)} \chi(p; q, r, s)[N^{(\kappa)}(p; q+r+s) - N^{(\kappa)\dagger}(-p; -q-r-s)] \Big), \\ x = \pm 13, \end{aligned} \tag{4.18}$$

with $N^{(\kappa)}(p; q+r+s)$ defined in (3.59),

$$\begin{aligned} E_{\frac{1}{2}(p+q+r+s)} = \frac{1}{\sqrt{2}} \sum_{\kappa=1}^{4} \Big(\sum_{x=\pm 13} e^{i\varepsilon(x)\rho} [N^{(\kappa)}(p+q+r+s, x) \\ + N^{(\kappa)\dagger}(-p-q-r-s, -x)] + \sum_{|l|=1}^{4} \chi(l; r_1, r_2, r_3) \\ \times [N^{(\kappa)}(-l|r_1+r_2+r_3) - N^{(\kappa)\dagger}(l|-r_1-r_2-r_3)] \Big). \end{aligned} \tag{4.19}$$

The $1/\sqrt{2}$ factors in (4.18) and (4.19) can be traced to the fact that, in our units, the roots $(p-q)$ have length $\sqrt{2}$, while the roots (p) and $\frac{1}{2}(p+q+r+s)$ are of unit length.

In the above equations $\kappa = 1, 2, 3, 4$, l is an element of the set p, q, r, s and r_1, r_2, r_3 are the remaining elements.

The new expressions which appear in (4.19) are

$$e^{i\rho} \equiv \tfrac{1}{2}(1+i\sqrt{3}), \tag{4.20}$$

$$N^{(\kappa)}(p+q+r+s, x) = a^{(\kappa)\dagger}(p+q+r+s)a^{(\kappa)}(x), \tag{4.21}$$

and

$$N^{(\kappa)}(-l\,|\,r_1+r_2+r_3) \equiv a^{(\kappa)\dagger}(-l+r_1+r_2+r_3)a^{(\kappa)}(-l), \qquad l, r_1, r_2, r_3 = \pm 1, \ldots, \pm 4. \tag{4.22}$$

The remaining phase factors $\varepsilon(p+q)$, $\psi(p, q; r, s)$ and $\chi(p; q, r, s)$ are identical to the phase factors ε, ψ and χ defined in connection with our previous discussions of D(4) and B(4) SHO realisations. Indeed,

$$\psi(p, q; r, s) \equiv f(p, q; r, s), \qquad \chi(p; q, r, s) \equiv \phi(p; q, r, s), \tag{4.23}$$

where f and ϕ were given in FFMa.

We note the appearance, for the first time, of a complex phase factor $e^{i\rho}$, associated with the occurrence of two zero weight elements in the irrep f_1 of F(4).

The scalar and f_1 irreps have the familiar forms, in analogy with the classical groups. The irrep $f_2 \equiv a$ is generated from the Kronecker product of two f_1's:

$$\begin{array}{llllll} [1_4]\oplus[1_4] & = [1_0] & \oplus[1_4] & \oplus[1_1] & \oplus[1_3] & \oplus[2_4] \\ f_1 \otimes f_1 & = s & \oplus f_1 & \oplus f_2 & \oplus A(2f_1) & \oplus S(2f_1) \\ 676 & = 1 & +26 & +52 & +273 & +324. \end{array} \tag{4.24}$$

All the other irreps can also be constructed from Kronecker products of f_1, but we will not explicitly display any of them.

4.3. E(6)

E(6) and E(7) are the simplest of the exceptional groups to treat. All of their non-vanishing roots are of the same length, and their lowest dimensional elementary irreps have no zero weight elements.

4.3.1. Realisation of the algebra. The simplest embedding of E(6) is to choose

$$D(27) \supset E(6), \tag{4.25}$$

with the relationships

$$\begin{array}{lllll} [1_1] & \to [1_1] & \oplus [1_5] & & \\ f & \to f_1 & \oplus f_2, & & f_2 \equiv \bar{f}_1. \\ 54 & \quad 27 & \quad \overline{27} & & \end{array} \tag{4.26}$$

It is convenient to introduce an unnormalised $\boldsymbol{\mu}_6$ basis vector (see FFMa):

$$\boldsymbol{\mu}_6 \cdot \boldsymbol{\mu}_6 = \tfrac{1}{3}, \qquad \boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_6 = 0, \qquad \boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_q = \delta_{pq}, \qquad p = 1, \ldots, 5. \tag{4.27}$$

In what follows, we will choose index labels $p, q \ldots, t = \pm 1, \ldots, \pm 5$ and $x = \pm 6$. The E(6) generators in terms of SHO operators are as follows.

The generators in $\boldsymbol{H}$, where

$$\boldsymbol{H} = \sum \boldsymbol{\mu}_p H(p) + \boldsymbol{\mu}_6[H(6) - H(-6)] \tag{4.28}$$

are

$$H(p) = \sum\nolimits'_{(p)} N^{(\kappa)}(p+x) + \tfrac{1}{2}\sum\nolimits'_{(p)} N^{(\kappa)}(p+q+r+s+t+x), \tag{4.29}$$

and

$$H(6) = \sum\nolimits'_{(6)} N^{(\kappa)}(p+6) + \sum\nolimits'_{(6)} N^{(\kappa)}(p+q+r+s+t+6) + 2\sum\nolimits'_{(6)} N^{(\kappa)}(2(6)), \tag{4.30}$$

with

$$N^{(\kappa)}(p+x) = a^{(\kappa)\dagger}(p+x)a^{(\kappa)}(p+x). \tag{4.31}$$

The generators corresponding to roots of non-vanishing weight, with real phase factors, and explicit solutions given, are

$$\begin{aligned} E_{(p-q)} = {} & \varepsilon(p+q) \sum\nolimits''_{(p,q)} \varepsilon(x)[N^{(\kappa)}(p+x, q+x) - N^{(\kappa)\dagger}(-p-x, -q-x)] \\ & + \sum\nolimits''_{(p,q)} \psi(p, q; r, s, t)[N^{(\kappa)}(p-q; r+s+t+6) \\ & - N^{(\kappa)}(p-q; -r-s-t-6)], \end{aligned} \tag{4.32}$$

$$\begin{aligned} E_{\{\frac{1}{2}(p+q+r+s+t)6(\pm)\frac{3}{2}(6)\}} = {} & \sum_{\kappa=1}^{5} [N^{(\kappa)}(p+q+r+s+t(\mp)⑥) \\ & + N^{(\kappa)\dagger}(-p-q-r-s-t(\pm)⑥)] + \sum_{\kappa=1}^{5} \sum_{l=1}^{5} \chi(l; r_1, r_2, r_3, r_4) \\ & \times [N^{(\kappa)}(-l \,|\, r_1+r_2+r_3+r_4(\pm)⑥) - N^{(\kappa)\dagger}(l \,|\, -r_1-r_2-r_3-r_4(\mp)⑥)], \\ & l, r_1, \ldots, r_4 = \pm 1, \ldots, \pm 5, \end{aligned} \tag{4.33}$$

with the upper (lower) signs taken together, and with the following minus sign parities of the indices:

$$\begin{array}{ll} \tfrac{1}{2}(p+q+r+s+t)(\pm)\tfrac{3}{2}(6) & \text{overall odd, including } 6 \\ N^{(\kappa)}(\ldots), a^{(\kappa)\dagger}(\ldots) & \text{overall even, including } 6. \end{array} \tag{4.34}$$

The following new $N^{(\kappa)}$'s and their definitions are

$$N^{(\kappa)}(p+x, q+x) = a^{(\kappa)\dagger}(p+x)a^{(\kappa)}(q+x), \tag{4.35}$$

$$N^{(\kappa)}(p+q+r+s+t(\mp)⑥) \equiv a^{(\kappa)\dagger}(p+q+r+s+t(\mp)6)a^{(\kappa)}(\mp)2(6)), \tag{4.36}$$

and

$$N^{(\kappa)}(-l \,|\, r_1+r_2+r_3+r_4(\pm)⑥) \equiv a^{(\kappa)\dagger}(-l+r_1+r_2+r_3+r_4(\pm)6)a^{(\kappa)}(-l(\mp)6). \tag{4.37}$$

We note that, in order to construct all irreps, we must take

$$\kappa = 1, \ldots, 5. \tag{4.38}$$

In many ways, $\boldsymbol{\mu}_6$ plays the role of the zero weights in B(n).

The phase factors ψ and χ are identical to those defined in connection with our previous discussion of D(5) and B(5) SHO realisations. Indeed,

$$\psi_{(o)}(p, q; r, s, t) \equiv e(p, q; r, s, t, \mp 6), \tag{4.39}$$

where e was given in FFMa.

4.3.2. Realisations of the irreps. The irreps have the familiar forms, in analogy with the classical groups. However, because of reality requirements, it was necessary to embed E(6) in a larger group in such a way that f of the larger group went into both f_1 and $\bar{f}_1$ of E(6) (see relation (4.26)). We stress this feature by explicitly exhibiting the f_1 and $\bar{f}_1$ irreps:

$$\begin{aligned} &|((\pm)\boldsymbol{\mu}_p - \boldsymbol{\mu}_6), [1_1]\rangle = a^{(\kappa)\dagger}((\pm)p-6)|0\rangle \\ &\left|(\tfrac{1}{2}\sum_{(e)}(\pm)\boldsymbol{\mu}_p + \boldsymbol{\mu}_6), [1_1]\right\rangle = a^{(\kappa)\dagger}(p+q+r+s+t+6)|0\rangle \\ &|2\boldsymbol{\mu}_6, [1_1]\rangle = a^{(\kappa)\dagger}(2(6))|0\rangle, \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} &|((\pm)\boldsymbol{\mu}_p + \boldsymbol{\mu}_6), [1_5]\rangle = a^{(\kappa)\dagger}((\pm)p+6)|0\rangle \\ &\left|(\tfrac{1}{2}\sum_{(o)}(\pm)\boldsymbol{\mu}_p - \boldsymbol{\mu}_6), [1_5]\right\rangle = a^{(\kappa)\dagger}(p+q+r+s+t-6)|0\rangle \\ &|-2\boldsymbol{\mu}_6, [1_5]\rangle = a^{(\kappa)\dagger}(-2(6))|0\rangle. \end{aligned} \tag{4.41}$$

The irrep $f_3 \equiv a$ is generated from the Kronecker product of f_1 and $\bar{f}_1 \equiv f_2$:

$$\begin{aligned} &[1_1]\otimes[1_5] = [0]\oplus[1_6]\oplus[1_1, 1_5] \\ &f_1 \otimes f_2 \quad = s \quad \oplus f_3 \quad \oplus (f_1 f_2) \\ &729 \qquad = 1 \quad +78 \quad +650. \end{aligned} \tag{4.42}$$

All other irreps can also be generated from Kronecker products.

4.4. E(7)

The simplest embedding of E(7) is

$$D(28) \supset E(7), \tag{4.43}$$

with the relations

$$\begin{aligned} &[1_1] \rightarrow [1_6] \\ &f \quad \rightarrow f_1 \\ &56 \qquad 56. \end{aligned} \tag{4.44}$$

As in the case of E(6), it is convenient to introduce an unnormalised basis vector (see FFMa), this time $\boldsymbol{\mu}_7$:

$$\begin{aligned} &\boldsymbol{\mu}_7 \cdot \boldsymbol{\mu}_7 = \tfrac{1}{2}, \\ &\boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_q = \delta_{pq}, \qquad \boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_7 = 0, \qquad p, q = 1, \ldots, 6. \end{aligned} \tag{4.45}$$

In what follows, we will choose index labels $p, q, \ldots, u = \pm 1, \ldots, \pm 6$ and $x = \pm 7$. The E(7) generators in terms of SHO operators are given as follows.
The generators in $\boldsymbol{H}$, where

$$\boldsymbol{H} = \sum \boldsymbol{\mu}_p H(p) + \boldsymbol{\mu}_7(H(7) - H(-7)) \tag{4.46}$$

are

$$H(p) = \textstyle\sum'_{(p)} N^{(\kappa)}(p+x) + \frac{1}{2}\sum'_{(p)} N^{(\kappa)}_{(o)}(p+q+r+s+t+u), \tag{4.47}$$

$$H(7) = \textstyle\sum'_{(7)} N^{(\kappa)}(p+7). \tag{4.48}$$

The generators corresponding to roots of non-vanishing weight, with real phase factors, and explicit solutions given, are

$$E_{(p-q)} = \varepsilon(p+q) \textstyle\sum''_{(p,q)} [N^{(\kappa)}(p+x, q+x) - N^{(\kappa)\dagger}(-p-x, -q-x)]$$
$$+ \textstyle\sum''_{(p,q)} \psi_{(e)}(p, q; r, s, t, u) N^{(\kappa)}_{(o)}(p-q; r+s+t+u), \tag{4.49}$$

$$E_{2(7)} = \textstyle\sum'_{(7)} [N^{(\kappa)}(p+7, p-7) - N^{(\kappa)\dagger}(-p-7, -p+7)], \tag{4.50}$$

the Hermitian conjugate of (4.50), and

$$E_{\{\frac{1}{2}(p+q+r+s+t+u)_{(e)(\pm)}(7)\}}$$
$$= \sum_{\kappa=1}^{6} \sum_{l=1}^{6} \xi_{(e)}((\pm)7 | l; r_1, \ldots, r_5)[N^{(\kappa)}_{(o)}(-l|(\pm)7|r_1 + \ldots + r_5)$$
$$- N^{(\kappa)\dagger}_{(o)}(l|(\mp)7|-r_1 - \ldots - r_5)],$$
$$l, r_1, \ldots, r_5 = \pm 1, \ldots, \pm 6. \tag{4.51}$$

We note that only even (odd) numbers of negative indices p, q, r, s, t, u appear in $\frac{1}{2}(p+q+r+s+t+u)_{(e)}$, $\xi_{(e)}((\pm)7|l; r_1 + \ldots + r_5)(N^{(\kappa)}_{(o)}(p-q; r+s+t+u)$, $a^{(\kappa)\dagger}_{(o)}(p-q+r\ldots))$. The new $N^{(\kappa)}$ which appears in (4.51) is defined as

$$N^{(\kappa)}_{(o)}(-l|(\pm)7|r_1 + \ldots + r_5) \equiv a^{(\kappa)\dagger}_{(o)}(-l + r_1 + \ldots + r_5) a^{(\kappa)}(-l(\mp)7). \tag{4.52}$$

We have not succeeded in obtaining simple algebraic expressions for the real phase factors $\psi_{(e)}$ and $\xi_{(e)}$ which appear in (4.49) and (4.51), respectively. However, they can easily be generated in tabular form from the following algebraic conditions, generated from the E(7) algebra, with E(7) phase solutions as given in FFMa:

$$\xi_{(e)}(7 | l; r_1, \ldots, r_5) = -\xi_{(e)}(-7 | -l; -r_1, \ldots, -r_5), \tag{4.53}$$

$$\xi_{(e)}(7 | -p; q, \ldots, u) \xi_{(e)}(7 | -q; p, \ldots, u) = \textstyle\prod''_{(p,7)} \varepsilon(|p| + l) \prod''_{(q,7)} \varepsilon(|q| + l'), \tag{4.54}$$

$$\xi_{(e)}(7 | l; r_1, \ldots, r_5) \xi_{(e)}(-7 | l; r_1, \ldots, r_5) = (-1)^{\Sigma l_-}, \tag{4.55}$$

$$\psi_{(e)}(p, q; r, s, t, u) = -\varepsilon(p+q) \xi_{(e)}(7 | p; q, \ldots, u) \xi_{(e)}(7 | q; p, \ldots, u), \tag{4.56}$$

where l_- are all possible values of negative indices. The tables can be generated, starting with the choice of a suitable set of $\xi_{(e)}$'s, and using (4.53)-(4.56) to obtain the remaining $\xi_{(e)}$'s and the $\psi_{(e)}$'s. (The latter are the phases $\psi_{(e)}$ which appear in the appropriate expressions for D(6) and B(6).) We have done so, but will not further encumber the present work by listing our specific results.

The scalar and f_1 irreps have the familiar forms, in analogy with the classical groups. The irreps $f_2 \equiv a$ and f_3 can be generated as follows

$$\begin{array}{llllll} f_2\colon [1_6]\otimes[1_6] & = [0] & \oplus[1_1] & \oplus[1_5] & \oplus[2_6] \\ f_1\otimes f_1 & = s & \oplus f_2 & \oplus A(2f_1) & \oplus S(2f_1) \\ 3136 & = 1 & +133 & +1539 & +1463, \end{array} \tag{4.57}$$

$$\begin{array}{llll} f_3\colon [1_6]\otimes[1_1] & = [1_6] & \oplus[1_7] & \oplus[1_1, 1_6] \\ f_1\otimes f_2 & = f_1 & \oplus f_3 & \oplus (f_1 f_2) \\ 7448 & = 56 & +912 & +6480. \end{array} \tag{4.58}$$

All other irreps can be generated from appropriate Kronecker products.

4.5. E(8)

The consideration of E(8) is straightforward, but algebraically tedious, largely because of the eight irrep elements of vanishing weight in $f_1 \equiv a$. Because of this fact, the solutions for the complex phase factors need to be given in tabular form. In the interests of brevity, we have only given the algebraic constraint equations for these phase factors.

The simplest embedding of E(8) is to choose

$$\mathrm{D}(124) \supset \mathrm{E}(8), \tag{4.59}$$

with the relations

$$\begin{array}{lll} [1_1] \to [1_1] & & \\ f \quad \to f_1, & & f_1 \equiv a. \\ 248 \quad 248 & & \end{array} \tag{4.60}$$

In what follows below, we will choose index labels $p, q, \ldots, w = \pm 1, \ldots, \pm 8$ and $z = 121, 122, 123, 124$ such that the $\boldsymbol{\lambda}$ space to $\boldsymbol{\mu}$ space projection of $\boldsymbol{\lambda}_z$ is $\boldsymbol{\lambda}_z \to 0$.

The E(8) generators in terms of SHO operators are given as follows.

The generators in $\boldsymbol{H}$, where

$$\boldsymbol{H} = \sum \boldsymbol{\mu}_p H(p) \tag{4.61}$$

are

$$H(p) = \sum\nolimits'_{(p)} N^{(\kappa)}(p-r) + \frac{1}{2}\sum\nolimits'_{(p)} N^{(\kappa)}_{(\mathrm{o})}(p+q+\ldots+w). \tag{4.62}$$

The generators corresponding to roots of non-vanishing weight, with most of the phase solutions explicitly given, are

$$\begin{aligned} E_{(p-q)} = {} & \frac{1}{2}\sum\nolimits''_{(p,q)} \omega(p-q; z)[N^{(\kappa)}(p-q, z) + N^{(\kappa)\dagger}(-p+q, -z)] \\ & + \sum\nolimits''_{(p,q)} d(p, q, r)[N^{(\kappa)}(p-r, -r+q) - N^{(\kappa)\dagger}(-p+r, r-q)] \\ & + \sum\nolimits''_{(p,q)} \psi_{(\mathrm{e})}(p, q; r, \ldots, w) N^{(\kappa)}_{(\mathrm{o})}(p-q; r+s+\ldots+w), \end{aligned} \tag{4.63}$$

with

$$N^{(\kappa)}(p-q, z) \equiv a^{(\kappa)\dagger}(p-q)a^{(\kappa)}(z),$$

and

$$\begin{aligned} E_{\frac{1}{2}(p+q+\ldots+w)} = \sum_{\kappa=1}^{8} \Bigg(& \frac{1}{2} \sum_{z=\pm 121}^{\pm 124} \exp[\mathrm{i}\zeta_{(\mathrm{o})}(p, \ldots, w; z)] \\ & \times [N_{(\mathrm{o})}^{(\kappa)}(p+q+\ldots+w, -z) + N_{(\mathrm{o})}^{(\kappa)\dagger}(-p-q-\ldots-w, z)] \\ & + \sum_{l_1, l_2} \psi_{(\mathrm{e})}(l_1, -l_2; r_1, \ldots, r_6) \\ & \times [N_{(\mathrm{o})}^{(\kappa)}(-l_1-l_2 | r_1+\ldots+r_6) - N_{(\mathrm{o})}^{(\kappa)\dagger}(l_1+l_2 | -r_1-\ldots-r_6)] \Bigg), \\ & l_1, l_2, r_1, \ldots, r_6 = \pm 1, \ldots, \pm 8, \end{aligned} \tag{4.64}$$

with

$$\zeta_{(\mathrm{o})}(p, \ldots, w; z) = \varepsilon(n_- - n_+)(-1)^z \tfrac{1}{6}\pi, \tag{4.65}$$

and

$$N_{(\mathrm{o})}^{(\kappa)}(-l_1-l_2 | r_1+\ldots+r_6) \equiv a_{(\mathrm{o})}^{(\kappa)\dagger}(-l_1-l_2+r_1+\ldots+r_6)a^{(\kappa)}(-l_1-l_2) \tag{4.66}$$

a generalisation of (4.22). In (4.65), $n_+(n_-)$ are the number of positive (negative) indices $p, q, \ldots, w$. The factors $\frac{1}{2}$ in (4.63) and (4.64) are introduced so that the remaining coefficients are phase factors†. The phase factor $d(p, q, r)$ is defined in (3.3) and $\psi_{(\mathrm{e})}$ first makes its appearance above in connection with D(8). Indeed,

$$\psi_{(\mathrm{e})}(p, q; r, \ldots, w) \equiv e(p, q; r, \ldots, w), \tag{4.67}$$

where e was given in FFMa.

We have not been able to obtain a general form for the remaining set of phase factors, $\omega(p-q; z)$. One can obtain a tabular form for them by generating them from the solutions for the set of simple roots. Since we are principally concerned in this paper with demonstrating the existence of SHO realisations of algebras and irreps, we will not give the details of the results, but will provide the equations constraining the ω's for simple roots.

A set of eight simple roots of E(8) can be taken to be, for any specific choice of $p, q, \ldots w$:

$$\begin{aligned} & \boldsymbol{\beta}_{\bar{1}} = \boldsymbol{\beta}_p^q, \quad \boldsymbol{\beta}_{\bar{2}} = \boldsymbol{\beta}_q^r, \quad \boldsymbol{\beta}_{\bar{3}} = \boldsymbol{\beta}_r^s, \quad \boldsymbol{\beta}_{\bar{4}} = \boldsymbol{\beta}_s^t, \quad \boldsymbol{\beta}_{\bar{5}} = \boldsymbol{\beta}_t^u, \\ & \boldsymbol{\beta}_{\bar{6}} = \boldsymbol{\beta}_{uvw}^{pqrst}, \quad \boldsymbol{\beta}_{\bar{7}} = \boldsymbol{\beta}_{pqrstuv}^{w}, \quad \boldsymbol{\beta}_{\bar{8}} = \boldsymbol{\beta}_u^v, \\ & p, q, \ldots, w = \pm 1, \ldots, \pm 8, |p| \neq |q| \neq \ldots \neq |w| \end{aligned} \tag{4.68}$$

with the Dynkin–Patera ordering, $\bar{j}$, $\bar{j} = 1, \ldots 8$, of simple roots and the definitions

$$\begin{aligned} & \boldsymbol{\beta}_p^q \equiv \boldsymbol{\mu}_p - \boldsymbol{\mu}_q, \\ & \boldsymbol{\beta}_{uvw}^{pqrst} \equiv \tfrac{1}{2}(-\boldsymbol{\mu}_p - \ldots - \boldsymbol{\mu}_t + \boldsymbol{\mu}_u + \boldsymbol{\mu}_v + \boldsymbol{\mu}_w), \text{ etc.} \end{aligned} \tag{4.69}$$

† This is the sole example of all the cases we have considered, in which equation (4.47) of FFMb, which arises from the commutators (our equation (2.3)), plays an essential role.

The conditions on the ω's associated with $\boldsymbol{\beta}_{\bar{1}}, \ldots \boldsymbol{\beta}_{\bar{5}}$ and $\boldsymbol{\beta}_{\bar{8}}$ are

$$\begin{aligned}
&\sum_z \omega(\bar{j}; z)\omega^*(\bar{k}; z) = 4\boldsymbol{\beta}_{\bar{j}} \cdot \boldsymbol{\beta}_{\bar{k}}, \\
&\sum_z \omega(\bar{j}; z) \exp[-\mathrm{i}(-1)^z \pi/6] = 4\delta_{\bar{j}\bar{5}}, \\
&\sum_z \omega(\bar{j}; z) \exp[\mathrm{i}(-1)^z \pi/6] = 0, \qquad \bar{j}, \bar{k} = \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{8}.
\end{aligned} \tag{4.70}$$

The scalar and $f_1 \equiv a$ irreps have the familiar form. The irreps f_2 and f_3 can be generated in a fashion analogous to the method used in E(7):

$$\begin{array}{lllllll}
f_2\colon [1_1]\otimes[1_1] & = [0] & \oplus[1_1] & \oplus[1_7] & \oplus[1_2] & \oplus[2_1] & \\
f_1 \otimes f_1 & = s & \oplus f_1 & \oplus f_2 & \oplus A(2f_1) & \oplus S(2f_1) & \quad (4.71) \\
61\,504 & = 1 & +248 & +3875 & +30\,380 & +27\,000 &
\end{array}$$

$$\begin{array}{lllllll}
f_3\colon [1_1]\otimes[1_7] & = [1_1] & \oplus[1_7] & \oplus[1_8] & \oplus[1_2] & \oplus[1_1, 1_7] & \\
f_1 \otimes f_2 & = f_1 & \oplus f_2 & \oplus f_3 & \oplus A(2f_1) & \oplus (f_1 f_2) & \quad (4.72) \\
961\,000 & = 248 & +3875 & +147\,250 & +30\,380 & -779\,247. &
\end{array}$$

All other irreps can also be generated from appropiate Kronecker products.

5. Summary and discussion

Our aim in the present work was to realise systematically and simply both the algebras and all irreps of all simple compact Lie groups in terms of Bose oscillators (SHO). In our uniform approach to all of these groups, the generators are bilinear expressions of sums of a^{+}'s and a's (SHO creation and annihilation operators) and the irreps are homogeneous polynomials of a^{+}'s of fixed order, operating on a vacuum state, of zero weight. In particular, the elementary spinor irreps of the orthogonal groups can be written in terms of a single SHO creation operator acting on the vacuum state. This is to be contrasted with the customary treatment of spinors in terms of Fermi oscillators (Clifford variables). In such a treatment the 'vacuum state' does not have zero weight, but is an element of a spinor irrep. The other elements of the irreps are obtained by having different powers of Fermi oscillator creation operators act on this 'vacuum state'. For D(n), the 'vacuum state', together with all states formed by even powers of a^{+}'s operating on this state, up to the maximum possible such power, constitute the elements of one of the elementary spinor irreps; all possible odd powers of a^{+}'s, operating on the 'vacuum', constitute the set of all elements of the other elementary spinor irrep.

Two features of our results, the possibility of realising all classical Lie algebras and their non-spinor irreps in terms of SHO's, and the need for the introduction of phase factors in the realisation of the algebras, are present in previous work. The novel features of the current work—the treatment of the exceptional groups and of spinor irreps of orthogonal groups—stem from a single ansatz: the systematic embedding of the group under consideration, g, in a larger group, $\mathscr{G}$, such that g is a non-regular subgroup of $\mathscr{G}$. $\mathscr{G}$ is always an orthogonal group. It is B(N) in the case of B(n) and G(2), and is D(N) in the case of all the other compact Lie groups, with appropriately chosen N's. Given the rank, n, of g, the rank, N, of $\mathscr{G}$, is fixed so that the elementary non-spinor irrep of $\mathscr{G}$ (called f) goes into those of the elementary irreps of g, from which all other irreps can be conveniently constructed through Kronecker products. When g is an orthogonal group, this requirement is met by having the f irrep of $\mathscr{G}$ go into the sum of all the elementary irreps of g; for the exceptional groups, the f irrep of $\mathscr{G}$ goes into the lowest dimensional irrep of g. For $\mathrm{g} = \mathrm{E}(6)$, there are two such irreps, complex conjugates of each other, and the f irrep of $\mathscr{G} = \mathrm{D}(27)$, which is real, goes into the sum of them.

The explicit solution of the embedding problem just outlined is made possible by two features of our approach. The first is the expression of root and weight space of all groups in terms of a set of mutually orthogonal basis vectors (Feldman *et al* 1984a, b, abbreviated FFMa and FFMb respectively). While such a choice of basis is a natural one in the case of the orthogonal groups, other, alternative, choices are frequently made for exceptional groups (Wybourne 1974, Dynkin 1957). The second element is the avoidance of the 'address' problem by the use of SHO operators, an issue which is discussed in connection with the presentation of the $\mathrm{D}(n)$ results above, but which we will now summarise.

In the embedding $\mathscr{G} \supset \mathrm{g}$, with g a non-regular subgroup, generators of type E_β in g are sums of generators $\mathscr{E}_\alpha$ of $\mathscr{G}$. The coefficients in these sums (both magnitudes and phases) depend on the structure constants of both $\mathscr{G}$ and g. This, in turn, requires that every element, $\boldsymbol{\lambda}$, of the root and weight space basis in $\mathscr{G}$ be given a single numerical 'address'. This requirement is obviated by the use of SHO operators. To be sure, such operators must be suitably labelled, and we label them in a one-to-one relation to the weights of the g irreps into which the f irrep of $\mathscr{G}$ goes. However, we know that the commutator $[a, a^\dagger]$ is unity only if the pair $a^\dagger$, a has the same label; otherwise it vanishes. An exact 'address' becomes irrelevant.

This feature of SHO operators enables us to exhibit explicit algebraic solutions for the embedding problem for orthogonal groups of arbitrary rank, and also for the exceptional groups. The embedding problem would, in principle, be soluble without the use of SHO operators, but would, in practice, be tedious and unmanageable. Indeed, an unforeseen consequence of the present work is that when maximal non-regular subgroups of a group are to be constructed, as is often the case when chains of symmetry breaking are considered in particle physics, the construction can be most simply carried out in the SHO picture.

The phase factors of the structure constants of g are, of course, still required to be explicitly known. There is, however, no 'address' problem in this case, and we can make use of the simple algebraic results for consistent solutions of these phase factors, obtained by FFMa. These are necessary to get the results presented above, as are the algebraic techniques elucidated by FFMb for carrying out the embedding of non-regular subgroups in a given group. A curious, and as yet not completely understood, feature of the current results is the reappearance of the phase solutions (and their obvious generalisations) obtained by FFMa, whenever phase factors with the same algebraic structure occur in the present work.

All the phase complications occur in the realisation of the generators. The elementary irreps of the classical groups are trivial in the SHO picture presented here, and are not much more complicated for the exceptional groups. The only additional prerequisite for constructing all basic irreps is the trivial introduction of an additional label (superscript κ) so that antisymmetric products of $a^\dagger$'s can be generated. This approach has been used in previous work.

No general expressions are presented for all irreps (not even for all basic irreps), but the tools for constructing all irreps are provided. We also present a sufficient number of specific illustrations, so that the reader can obtain a particular irrep as needed.

There is considerable redundancy in the irreps we construct. We have to eliminate some of them by means of 'SHO reduction', as discussed at the end of § 3.1. For example, there is a redundant scalar irrep, which arises from a reduction of a product of two $a^{\dagger}$'s. It is in the form of a homogeneous quadratic polynomial of $a^{\dagger}$'s, operating on the vacuum, and is 'SHO reduced' to the standard scalar, the vacuum state. Other, more trivial, redundancies, associated with different sets of (κ) superscripts, remain, but could be eliminated by a simple ordering ansatz. There also exist alternative realisations of the algebras and therefore of the irreps. Two examples are provided by the groups E(7) and E(6), because of the regular maximal subgroup relations $\mathrm{E}(8) \supset \mathrm{E}(7) \otimes \mathrm{A}(1)$ and $\mathrm{E}(7) \supset \mathrm{E}(6) \otimes \mathrm{U}(1)$. Realisations of the E(7) and E(6) generators, alternative to the ones we give above, can be obtained by taking appropriate subsets of the E(8) generators exhibited in § 4.5. Correspondingly, we obtain alternative realisations of the irreps of these groups. In particular, the adjoint irreps are linear, rather than quadratic in the $a^{\dagger}$'s in this approach. However, we feel that the realisations of the algebras and irreps of E(7) and E(6) we present are simpler than these alternatives.

In any case, there is at least one way of realising each irrep in terms of SHO operators, including irreps with spinor content. Since the SHO operators $a^{\dagger}$ and a, as well as the SHO ground state (our vacuum state $|0\rangle$), can easily be expressed in the coordinate basis, in terms of suitable coordinates and their derivatives, the realisations of the algebra and the other irreps can also be so expressed.

Acknowledgments

The author would like to thank the Theory Division of CERN, where most of this work was carried out, for its hospitality in the academic year 1983-84, and the Physics Department of the University of Florence, where some of the work was done in a brief visit in the autumn of 1983. He would also like to thank Susan Kövesi-Domokos, Gábor Domokos and Robert Casalbuoni for helpful comments. He particularly wishes to express his appreciation to Gordon Feldman for his customary incisive comments and helpful long-distance criticisms.

References

Alhassid T, Gürsey F and Iachello F 1983a *Phys. Rev. Lett.* **50** 873
—— 1983b *Ann. Phys., NY* **148** 346
Carter R W 1972 *Simple Groups of Lie Type* (New York: Wiley-Interscience) p 55
Casalbuoni R 1980 in *Developments in the Theory of Fundamental Interactions* (*Proc. 17th Winter School of Theor. Phys., Karpacz, Poland*) ed L Turko and A Pekalski (Chur: Harwood Academic) pp 267-89
Casalbuoni R and Gatto R 1980 *Phys. Lett.* **90B** 81
Dynkin E B 1950 *Am. Math. Soc. Transl.* **17**
—— 1957 *Am. Math. Soc. Transl. Ser. 2* **6** 111, 245
Dyson F J 1956 *Phys. Rev.* **102** 1217, 1231
Exner P, Havlíček M and Lassner W 1976 *Czech. J. Phys.* B **26** 1213

Feldman G, Fulton T and Matthews P T 1982 *J. Phys. G: Nucl. Phys.* **8** 295
—— 1984a *J. Math. Phys.* **25** 1222
—— 1984b *J. Math. Phys.* **25** 1230
Gilmore R 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley Interscience) pp 293-6
Judd B R 1967 *Second Quantization and Atomic Spectroscopy* (Baltimore, MD: The Johns Hopkins Press)
—— 1968 in *Group Theory and its Applications* ed E M Loebl (New York: Academic) pp 183-219
Kramer P and Moshinsky M 1968 in *Group Theory and Its Applications* ed E M Loebl (New York: Academic) pp 339-468
Lohe M A 1973 *J. Math. Phys.* **14** 1959
Lohe M A and Hurst C A 1971 *J. Math. Phys.* **12** 1882
—— 1973 *J. Math. Phys.* **14** 1955
McKay W G and Patera J 1981 *Tables of Dimensions, Indices and Branching Rules for Representations of Simple Lie Algebras* (New York: Marcel Dekker)
McKay W G, Patera J and Sankoff D 1977 in *Computers in Nonassociative Rings and Algebras* ed R E Beck and B Kalmar (New York: Academic)
Quesne C 1973 *J. Math. Phys.* **14** 366
Schwinger J 1965 *Quantum Theory of Angular Momentum* ed L Biedenharn and E van Dam (New York: Academic) pp 229-79
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)